Static and dynamic epidemia on chains and trees with two- and three-dimensional loops

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The dynamic epidemic model considers the expansion of a cluster in a medium containing a fraction x of mobile particles that are pushed by a propagation front. This model is exactly solved here on two- and three-dimensional chains and a Bethe tree, which are all decorated with consecutive either hexagon or tetrahedron loops. The exact values for the percolation threshold x_c and the critical exponents are calculated and compared to the static hindrance cases. The fraction of site candidates for particle trapping on a tree is the relevant parameter for the threshold value of such dynamic epidemics in high dimensions. [S1063-651X(98)09603-2]

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I. INTRODUCTION

The dynamic epidemic model is relevant for describing the spreading of a fluid in a medium containing mobile impurities and also other physical systems such as river spreading, traffic jams, rough surfaces, electrical conduction in heterogeneous media, and polymer growth in solutions. Vandewalle and Ausloos [1] have solved exactly such a dynamic epidemic model with the ''growth–transfer-matrix'' method for various chains and trees that contain loops and thus allow for trapping of particles along (and behind) the propagation front. The pushing and trapping of particles in such a model lead to nontrivial surface roughness [2,3].

More complicated basic units and higher-dimensional cases are of interest. It is indeed known that critical exponents are dimensionality dependent [4]. Higher-dimensional loops are also encountered in other fields of science such as biology, traffic, and economics. Finally, in higher dimensions, one can expect better agreement with theoretical considerations based on a Bethe lattice [5], which can be considered as an infinite-dimensional network.

The aim of this paper is to provide an exact solution of the static and dynamic epidemics model on lattices with loops constructed in a plane or in three-dimensional (3D) space. A chain of hexagons, a 3D chain of tetrahedrons, and a tree of tetrahedrons are considered below. Notice that the chain with hexagons generalizes the problem studied in Ref. [1] already as if a multiple invasion step or several stages can occur. In traffic physics this can be considered to correspond to different by-passing velocities, or stops at parking places, in crystal growth to different critical radii for trapping and consequently different ''critical velocities'' [6] and finally in fluid invasion to different retention times.

II. STATIC AND DYNAMIC EPIDEMIA ON LOOPED CHAINS

In Fig. 1 two chains are presented: The first one is a chain decorated with hexagons [Fig. 1(a)] and the second chain is one with tetrahedrons as loops between sites [Fig. 1(b)]. The empty sites are denoted by open circles, while the sites occupied by a particle are denoted by solid circles. The fraction φ of sites that are candidates for particle trapping has a different value on each lattice. For the chain looped with hexagons $\varphi=\frac{4}{5}$, while $\varphi=\frac{2}{3}$ for the chain looped with tetrahedrons. Let a fraction x of sites be occupied by particles and for convenience let the front go from left to right. At each growth step, if a nearest-neighboring site on the right is empty, the site is invaded. However, if this site contains a particle, the particle is pushed onto the next-nearest-neighboring site *if* the latter is empty. If this next-nearest-neighboring site contains a particle, the growth is blocked. When there is some bypassing channel, like in decorated chains, the percolation process is expected to take place up to some impurity concentration to be determined.

A. Chain of tetrahedrons

The basic set of local configurations on a chain of tetrahedrons contains four local sites (Fig. 2). In order to describe the dynamical process of cluster spreading and particle pushing on a chain looped with tetrahedrons, there are $2^4=16$ different possible configurations to examine (Fig. 2). As a result, a 16×16 transfer matrix T has to be written corresponding to an invasion front from the left-hand site of the chain. As was pointed out in Ref. [1], the configuration label or ''configuration order'' is crucial for the optimal description of the transfer matrices. Each element of the T matrix T_{ij} is the probability of obtaining the configuration labeled i invading the configuration labeled j such that $|i\rangle=\Sigma T_{ij}|j\rangle$. Several configurations carry identical weight, but for a better structure of the matrix and computational precision all 16 elements have been kept at first. The evolution rules and the techniques of retrieving the T_{ij} are described in detail in Ref. [1] for the chain of triangles. The same procedure has been followed here. The nonzero elements of the T matrix when the epidemics invade the left-hand site of each configuration (Fig. 2) can be easily written and are omitted here due to space limitation.

One has to know the probability of finding the next configuration on the chain, i.e., the growth evolution matrix G made of two vector rows

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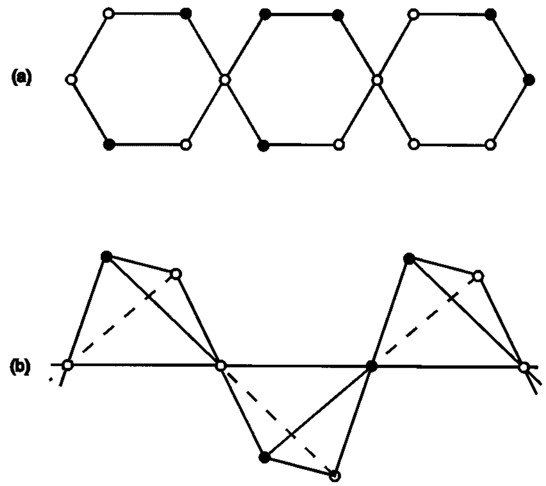

FIG. 1. Two different types of chains: (a) a chain of hexagons and (b) a chain of tetrahedrons. The empty sites are denoted by open circles, while the sites occupied by a particle are denoted by solid circles.

$$s_1=\{1,1,1,1,0,1,1,0,1,0,0,1,0,0,0,0\}$$

$$s_2=\{0,0,0,0,1,0,0,1,0,1,1,0,1,1,1,1\}.$$

Thus the G matrix can be constructed as each element of the row is multiplied by x^3, $x^2(1-x)$, $(1-x)^2$, or $(1-x)^3$ in the following manner: 1st row, $s_1(1-x)^3$; 2nd row, $s_2(1-x)^3$; 3rd to 5th rows, $s_1x(1-x)^2$; 6th to 8th rows, $s_2x(1-x)^2$; 9th to 11th rows, $s_1x^2(1-x)$; 12th to 14th rows, $s_2x^2(1-x)$; 15th row, s_1x^3; 16th row, s_2x^3, where each element G_{ij} is the average number of configurations i next to a configuration j. The sum of the elements of each column of G is also 1 since each local configuration is connected to only one configuration on a chain. The growth process is thus well described by the iteration of the matrix GT. This matrix has two degenerate nonzero eigenvalues $\lambda_\pm$ given by

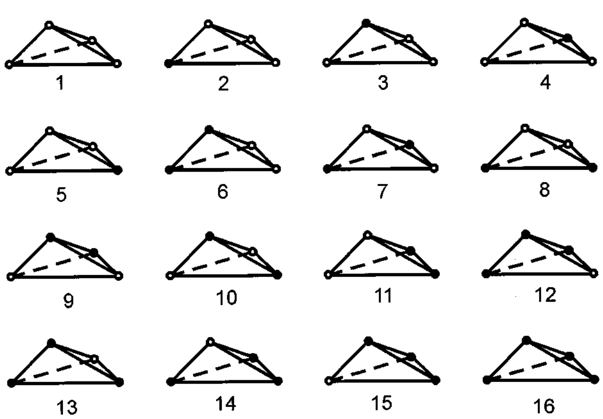

FIG. 2. Sixteen local configurations used to describe the dynamic epidemics model on a chain of tetrahedrons. The empty sites are denoted by open circles, while the sites occupied by a particle ar denoted by solid circles.

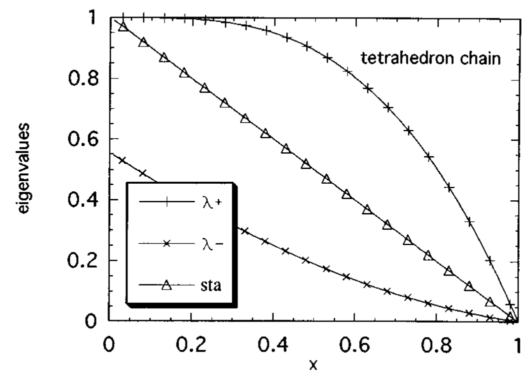

FIG. 3. Nonzero eigenvalues λ_+ and λ_- of the growth evolution transfer matrix GT as a function of the fraction x of particles for a chain of tetrahedrons. The eigenvalue for the static case is also illustrated.

$$\lambda_\pm=\tfrac{1}{9}[7-4x+2x^2-5x^3\pm(1-x)\times(4+24x+52x^2+84x^3+7x^4)^{1/2}], \qquad (1)$$

which are both strictly less than 1 expect for $x=0$ (Fig. 3). The critical point $x_c^{(\mathrm{dyn})}$ occur when the largest eigenvalue is equal to 1. The spreading of the invading cluster through the chain is thus possible only in the absence of impurities since $x_c^{(\mathrm{dyn})}=0$.

One should note that the case of static particles is recovered for a T matrix whose nonzero elements are (1,1), (3,3), (4,4), and (9,9) only; each such element is equal to one. Thus the $GT^{(\mathrm{sta})}$ matrix has only one nonzero somewhat trivial eigenvalue $\lambda=1-x$ (Fig. 3) giving $x_c^{(\mathrm{sta})}=0$.

B. Chain of hexagons

On a chain of hexagons, the basic set of local configurations contains six sites. Thus $2^6=64$ different possible configurations are needed to describe the epidemia propagation on the chain with hexagon loops. A three-stage invasion has to be considered due to existing configurations with two sites on the top and/or bottom row being candidates on which the particle trapping can take place. The transfer T and growth G matrices can be easily constructed. It has been numerically estimated that two nonzero eigenvalues λ_+ and λ_-exist for this dynamic epidemics. These eigenvalues are shown in Fig. 4 as a function of the particle fraction x. The critical fraction of particles is $x_c^{(\mathrm{dyn})}=0$. One can easily observe that the corresponding $GT^{(\mathrm{sta})}$ for static particles has only one nonzero eigenvalue λ (Fig. 4). The critical threshold for the static model is $x_c^{(\mathrm{sta})}=0$. One should note the linear dependence of the eigenvalue λ corresponding to the static case on the fraction x, in contrast to the one-dimensional chain looped with squares.

C. Critical exponents

Various physical quantities usually exhibit a power-law divergence near a percolation threshold; such are the linear

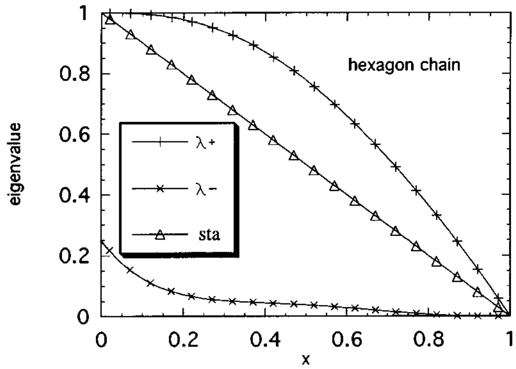

FIG. 4. Nonzero eigenvalues λ_+ and λ_- of the growth-evolution transfer matrix GT as a function of the fraction x of particles on a chain of hexagons. The behavior of the eigenvalue for the static case is also given.

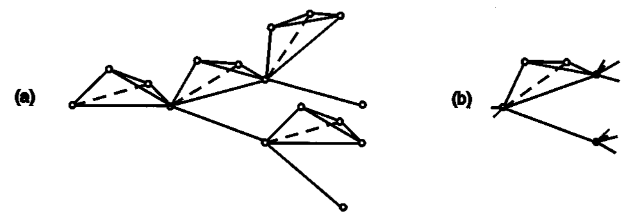

FIG. 5. Hierarchical tree with a branching rate $z=2$: (a) one branch over two is looped with a tetrahedron and (b) a small portion of the tree is used as a local configuration for studying the dynamic epidemics model.

size ξ of the spreading cluster and the cluster mass S, $\xi \sim (x-x_c)^{-\nu}$ and $S \sim (x-x_c)^{-\gamma}$ above x_c thereby defining the basic critical exponents ν and γ. Assuming that the largest eigenvalue drives the growth process, the mean number $g(r)$ of sites lying on the spreading cluster at a distance r from the initial site is also obtainable from $g(r) \sim \lambda_+^r$.

Developing $1-\lambda_\pm$ in powers of x around $x_c=0$, one can obtain the scaling laws $\xi^2(1-\lambda_+)^{-2}$ and $S \sim (1-\lambda_+)^{-1}$ and thus retrieve the critical exponents ν and γ. One should notice that $\nu=\gamma$ for chains [1].

First, for the chain looped with tetrahedrons the transition is characterized by $\nu=\gamma=1$ in the static case since $\lambda=1-x$. The expansion of λ_+ [from Eq. (3)] gives $\nu=\gamma=3$ for the dynamic epidemics.

Next, the behavior of λ_+ and λ_- has been numerically studied near the critical $x_c^{(\mathrm{dyn})}=0$ for the mobile particles placed on a chain looped with hexagons. It is found that for dynamic epidemics on such a chain $\nu=\gamma=3$, the same exponent as that found in Ref. [1] for the chain of squares and that of triangles. The eigenvalue for the static model of the hexagonal chain has been numerically obtained to be equal to $1-x$. Thus $\nu=\lambda=1$ for the static case.

III. STATIC AND DYNAMIC EPIDEMIA ON A BETHE TREE WITH TETRAHEDRON LOOPS

The Bethe tree with tetrahedron loops is obtained from the usual Bethe lattice with a branching rate $z=2$ for which one branch out of two is a tetrahedron (corresponding to the type-I tree of triangles in Ref. [1]). One can also consider that the Bethe lattice has locally $z=2$ or 4 every two nodes. The local configurations to be examined contain five sites as schematically drawn in Fig. 5. The fraction of trapping sites is $\varphi=\frac{2}{3}$. In order to describe the dynamic epidemics model on such a tree, there are $2^5=32$ different possible configurations. The T and G matrices can be constructed considering the two-stage invasion of the propagation front. One should emphasize that the sum of the elements of each column of G is now equal to 2 instead of 1 as in Sec. II for chains since each local configuration is followed by $z=2$ next possible ones.

Even though it was possible to write down the 32×32 T and G matrices, taking into account symmetric configurations, it was not possible to extract analytically the eigenvalues of the GT matrix. It is found numerically that two different eigenvalues have nonzero values for the dynamic epidemics. Similarly to previous results, one calls then λ_+ for the largest one and λ_- for the second one. These eigenvalues are plotted in Fig. 6 as a function of x. As x increases from zero to 1, λ_+ decreases from 2 to zero. It is found numerically that $x_c^{(\mathrm{dyn})}=0.851\,64(1)$ for trees looped with tetrahedrons. This value is a little bit larger than the critical value 0.812 found numerically for trees of type I (looped with triangles) in Ref. [1], but very close to the critical value 0.849 for trees of type II having the same fraction of sites $\varphi=\frac{2}{3}$ that are possible candidates for particle trapping. Recall that the percolation threshold for ordinary Bethe lattices is $x_c^{(\mathrm{dyn})}=(z^2-1)/z^2$ in the dynamic case, but $x_c^{(\mathrm{sta})}=(z-1)/z$ in the static case. One easily verifies that the tree decorated with tetrahedrons corresponds to a tree with an effective branching ratio equal to 2.596. Moreover, it can be numerically analyzed how λ_+ reaches 1 near $x_c^{(\mathrm{dyn})}$ (from below). It is found that $1-\lambda_+$ scales as $x-x_c^{(\mathrm{dyn})}$. Thus $\nu=\gamma=1$ for the dynamic epidemia on such a tree of tetrahedrons.

For the static impurity case the T matrix is reduced to a diagonal $T^{(\mathrm{sta})}$ matrix with 16 nonzero elements, but it was not possible to obtain analytically the $GT^{(\mathrm{sta})}$ matrix eigenvalues. However, it has been found numerically that there is only one nonzero eigenvalue $\lambda=2(1-x)$, which is drawn in Fig. 6. Thus, $x_c^{(\mathrm{sta})}=\frac{1}{2}$, corresponding to the static epidemics case on the usual Bethe lattice. This gives also $\nu=\gamma=1$.

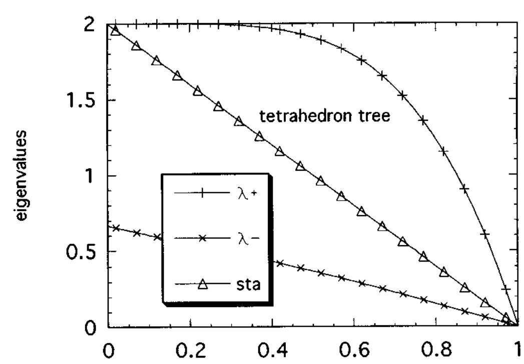

FIG. 6. Nonzero eigenvalues λ_+ and λ_- of the growth-evolution transfer matrix GT as a function of the fraction x of particles on a tree of tetrahedrons. The eigenvalue for the static case is also shown.

IV. DISCUSSION

A chain looped with hexagons, a spatial chain looped with tetrahedrons, and a tree looped with tetrahedrons have been used in order to describe dynamic invasion percolation in a medium containing mobile or stationary ''impurities.'' The epidemics front growth has been described through a growth-evolution–transfer-matrix method. The largest eigenvalue of the epidemia evolution matrix GT is a measure of the correlation length of the growing system. The linear dependence for the chain decorated with hexagons is unexpected if compared to the nonlinear behavior of the static eigenvalue for the chain with squares. It has been found that the critical exponents describing the divergence of the cluster mass S and correlation length ξ near the threshold x_c have the same values for both static and dynamic epidemics on a tree looped with tetrahedrons and are similar to those examined in Ref. [1].

On the two studied chains, the percolation thresholds for both static and dynamic epidemics are equal to zero. However, nonzero threshold values have been found on the 3D tree. The 3D tree and the tree of type II looped with triangles have the same fraction of trapping sites in fact. This leads one to emphasize the relevance of the fraction of sites that are candidates for particle trapping rather than the connectivity itself on the threshold value of dynamic epidemics. The theoretical results found herein are summarized in Table I. One should note that the universality of dynamic epidemics is recovered for the tree looped with tetrahedrons. Moreover, the values of the critical exponents for $d>1$ are found to be unchanged between static and dynamic epidemics. The behavior is again ''superuniversal'' for $d>1$, in contrast to $d=1$ lattices.

TABLE I. Percolation thresholds $x_c^{(\mathrm{sta})}$ and $x_c^{(\mathrm{dyn})}$ for, respectively, static and mobile particles on 1D-like lattices decorated with tetrahedrons or hexagons and on a Bethe tree with tetrahedrons. The fraction φ of sites that are possible candidates for trapping and critical components γ and ν describing the divergence of the cluster mass and the correlation length, respectively, near the threshold for static and dynamic epidemia are given.

Lattice	φ	$x_c^{(\mathrm{sta})}$	$x_c^{(\mathrm{dyn})}$	$\nu=\gamma$ sta	$\nu=\gamma$ dyn
chains of hexagons	$\frac{4}{5}$	0	0	1	3
chain of tetrahedrons	$\frac{2}{3}$	0	0	1	3
tree with tetrahedrons	$\frac{2}{3}$	0.5	0.851 64	1	1

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